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Reduction of System Order Using Power Spectral Density Function-Generalization of Liaw's Dispersion Analysis

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The method of model reduction based on dispersion analysis and the continued fraction is extended to treat the system which has multiple poles or has simple or multiple poles on the imaginary axis. Using the power spectral density function and preserving the dynamic modes with large power contibutions, the denominator of the reduced model is obtained and its numerator is obtained by using the continued-fraction method. This method is proved to give better approximation to an original system through examples than other methods.

Key Words: Order Reduction, Power Spectral Density Function, Continued Fraction Method

1. Introduction

Reduction of the system order enables one to simplify the design and analysis of high-order linear system. The method of reduction of system order using dispersion analysis(Liaw et al., 1986) is known to be more prominent than other methods(Shamash, 1975; Chen et al., 1980). In this method, the denominator of the reduced model is determined from the viewpoint of energy contribution to the system output; the dynamic modes (eigenvalues) with dominant energy contributions are preserved. In order to give each dynamic mode equal weighting, input-exciting signals are assumed to be white noises which are constant for frequencies. Therefore, the total power of each dynamic mode can be obtained by intergrating the power spectral density function over entire frequency range. By preserving the dynamic modes with large power, the denominator of the reduced model is obtained. Its numerator can be found by using the continued-fraction method.

The main disadvantage of this method is its inability in treating the system with multiple poles, or simple or multiple poles on the imaginary axis of Laplace domain. Therefore, the method is extended to overcome its inability in the present work.

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2. Determination of the Denominator Using Power Spectrum

To simplify the analysis, it is assumed that the denominator of the transfer function has multiple poles p_1, p_2 of multiplicities m and r, respectively, and other poles are distinct.

The n-th order transfer function $G(s)$ is given as

$$G(s)=\frac{B(s)}{A(s)}=\frac{B_1+B_2s+\cdots+B_ns^{n-1}}{A_1+A_2s+\cdots+A_{n+1}s^n}=\sum_{i=1}^{m}\frac{a_i}{(s+p_1)^i}+\sum_{i=1}^{r}\frac{b_i}{(s+p_2)^i}+\sum_{i=m+r+1}^{n}\frac{c_i}{(s+p_i)}, \quad (1)$$

where

$$a_{m-j}=\frac{1}{j!}\left\{\frac{d^j}{ds^j}\left[\frac{B(s)}{A(s)}(s+p_1)^m\right]\right\}_{s=-p_1},\quad j=0,1,\cdots,m-1$$

$$b_{r-j}=\frac{1}{j!}\left\{\frac{d^j}{ds^j}\left[\frac{B(s)}{A(s)}(s+p_2)^r\right]\right\}_{s=-p_2},\quad j=0,1,\cdots,r-1$$

$$c_j=\left[\frac{B(s)}{A(s)}(s+p_j)\right]_{s=-p_j},\quad j=m+r+1,m+r+2,\cdots,n.$$

The unit impulse response is obtained from Eq. (1):

$$G(t)=\sum_{i=1}^{m}\frac{a_i}{(i-1)!}t^{i-1}e^{-p_1t}+\sum_{i=1}^{r}\frac{b_i}{(i-1)!}t^{i-1}e^{-p_2t}+\sum_{i=m+r+1}^{n}c_ie^{-p_it}. \quad (2)$$

Through Fourier transformation applied in the case of deterministic power and energy signals, a one-to-one mapping between time and frequency domains is established. The power spectral density function and auto-correlation function are related for stationary signals. If one-sided power spectral density function is used, integration is carried out only over positive frequencies(Bendat et al., 1986):

$$R_{yy}(\tau)=\int_0^\infty G_{yy}(f)\cos(2\pi f\tau)\ df. \quad (3)$$

where $R_{yy}(\tau)$ is the auto-correlation function of the output and $G_{yy}(f)$ the power spectral density function of the output.

In particular, at $\tau=0$, we obtain

$$\int_0^\infty G_{yy}(f)\ df=R_{yy}(0). \quad (4)$$

Therefore, without solving the power spectral density function, we obtain the energy contribution of each dynamic mode. Input-exciting signals are assumed to be white noises η(t) in order to give each dynamic mode equal weighting.

The response of system is

$$y(t)=\int_{-\infty}^{t}G(t-v)\eta(v)dv. \quad (5)$$

Integrating the power spectrum over frequncies gives

$$\int_0^\infty G_{yy}(f)df=R_{yy}(0)=E\{y(t)y(t)\}. \quad (6)$$

Substituting Eq. (5) into Eq. (6) produces

$$\int_0^\infty G_{yy}(f)df=E\Big\{\int_{-\infty}^{t}G(t-v')\eta(v')dv'\int_{-\infty}^{t}G(t-v)\eta(v)dv\Big\}$$
$$=\int_{-\infty}^{t}\int_{-\infty}^{t}G(t-v)G(t-v')E\{\eta(v)\eta(v')\}dvdv'$$
$$=\sigma_\eta^2\int_{-\infty}^{t}G^2(t-v)dv$$
$$=\sigma_\eta^2\int_0^\infty G^2(t)dt$$
$$=\sigma_\eta^2\int_0^\infty\Big[\sum_{i=1}^{m}\frac{a_i}{(i-1)!}t^{i-1}e^{-p_1t}+\sum_{i=m+r+1}^{n}c_ie^{-p_it}\Big]^2dt, \quad (7)$$

where the auto-correlation of the white noise, $E\{\eta(v)\eta(v')\}=\sigma_\eta{}^2$.

Note that $\int_0^\infty t^ne^{-at}dt=\frac{\Gamma(n+1)}{a^{n+1}}$, where Γ means the gamma function. Eq. (7) is propagated as follows:

$$\int_0^\infty G_{yy}(f)\ df$$
$$=\sigma_\eta{}^2\Big\{\Big[\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{a_ia_j}{(i-1)!(j-1)!}\frac{\Gamma(i+j-1)}{(2p_1)^{i+j-1}}+\sum_{i=1}^{r}\sum_{j=1}^{r}\frac{b_ib_j}{(i-1)!(j-1)!}\frac{\Gamma(i+j-1)}{(2p_2)^{i+j-1}}+\sum_{i=m+r+1}^{n}\sum_{j=m+r+1}^{n}\frac{c_ic_j}{p_i+p_j}$$
$$+2\sum_{i=1}^{m}\sum_{j=1}^{r}\frac{a_ib_j}{(i-1)!(j-1)!}\frac{\Gamma(i+j-1)}{(p_1+p_2)^{i+j-1}}+2\sum_{i=1}^{r}\sum_{j=m+r+1}^{n}\frac{b_ic_j}{(i-1)!}\frac{\Gamma(i)}{(p_2+p_j)^i}+2\sum_{i=m+r+1}^{n}\sum_{j=1}^{m}\frac{c_ia_j}{(j-1)!}\frac{\Gamma(j)}{(p_1+p_i)^j}\Big]\Big\}. \quad (8)$$

The term, which contains $a_i(i=1,\cdots,m)$, $b_i(i=1,\cdots,r)$, and $c_i(i=m+r+1,\cdots,n)$, respectively, represents the importance of each dynamic mode. For example, the power contributions, $\{PC(a_1)\}$ and $\{PC(a_m)\}$, of the dynamic modes, $a_1(=a_1/s+p_1)$ and $a_m(=a_m/s+p_i)^m$, respectively, are as follows :

$$PC(a_1)=\sum_{i=1}^{m}\frac{a_1a_i}{(i-1)!}\frac{\Gamma(i)}{(2p_1)^i}+\sum_{i=1}^{r}\frac{a_1b_i}{(i-1)!}\frac{\Gamma(i)}{(p_1+p_2)^i}+\sum_{i=m+r+1}^{n}\frac{a_1c_i}{(p_1+p_i)},$$

$$PC(a_m)=\sum_{i=1}^{m}\frac{a_ma_i}{(m-1)!(i-1)!}\frac{\Gamma(i+m-1)}{(2p_1)^{i+m-1}}+\sum_{i=1}^{r}\frac{a_mb_i}{(m-1)!(i-1)!}\frac{\Gamma(i+m-1)}{(p_1+p_2)^{i+m-1}}+\sum_{i=m+r+1}^{n}\frac{a_mc_i}{(m-1)!}\frac{\Gamma(m)}{(p_1+p_i)^m}.$$

Neglecting the dynamic modes that have small power contribution of all dynamic modes corresponding to a multiple pole, there is a possibility of reducing the system order. As it were, if the importance of the dynamic mode $a_m(=a_m/(s+p_1)^m)$ of a multiple pole p_{1i} is relatively small in comparison with that of other dynamic modes a_i $(=a_i/(s+p_1)^i)$, $i=1,2,\cdots,(m-1)$, the multiplicity of the multiple pole p_1 can be reduced from m to $(m-1)$. The relatvie importance of each dynamic mode is estimated in terms of the ratio of its power contribution to the total power.

Since the power contribution of a complex pole due to Eq. (8) is a complex number which is meaningless and also, has a complex conjugate, its power contribution is defined as the sum for each conjugate pair of a complex pole in order to be a meaningful real number.

Equation (8) does not provide solutions for the transfer function which has simple or multiple poles at the imaginary axis of Laplace domain. Considering the transfer function that has a multiple pole with multiplicity m at the origin and of which other poles are distinct, the transfer function of this system is fractionated partially as follows:

$$G(s)=\sum_{i=1}^{m}\frac{a_i}{s^i}+\frac{B'(s)}{A'(s)}=\sum_{i=1}^{m}\frac{a_i}{s^i}+G'(s) \quad (9)$$

where

$$\frac{B'(s)}{A'(s)}=\sum_{i=1}^{n'}\frac{c_i}{s+p_i}.$$

Therefore, applying $G'(s)$ that the dynamic mode of the pole, zero, is removed, to Eq. (8) we can find the power contribution of each dynamic mode.

3. Determination of Nurmerator Using Continued Fraction Method

Since it does not matter whether a pole is simple or not in determining the numerator, we will consider the retained dynamic modes be $-p_1$, $-p_2$, $\cdots$, $-p_l$. The reduced model is as follows:

$$R(s)=\frac{B_{2,1}+B_{2,2}s+\cdots+B_{2,l}s^l}{(s+p_1)(s+p_2)\cdots(s+p_l)}$$
$$=\frac{B_{2,1}+B_{2,2}s+B_{3,1}s^2+\cdots+B_{2,l}s^{l-1}}{B_{1,1}+B_{1,2}s+B_{1,3}s^2+\cdots+B_{1,l+1}s^l} \quad (10)$$

where $B_{1,1}$, $B_{1,2}$,$\cdots$,$B_{1,l+1}$ are known and $B_{2,1}$, $B_{2,2}$, $\cdots$,$B_{2,l}$ can be found by matching the time moments.

Equation (1) is rewritten as

$$G(s)=\frac{A_{2,1}+A_{2,2}s+A_{2,3}s^2+\cdots+A_{2,n}s^{n-1}}{A_{1,1}+A_{1,2}s+A_{1,3}s^2+\cdots+A_{1,n+1}s^n}. \quad (11)$$

The continued fraction expansion of Eq. (11) about $s=0$ and $s=\infty$ has the following form:

$$G(s)=\cfrac{1}{h_1+\cfrac{s}{h_2+\cfrac{s}{h_3+\cfrac{s}{\cdots}}}} \quad (12)$$

The coefficients h_i are obtained from the coefficients $A_{j,k}$ of Eq. (11) by forming a Routh array(Bosley et al., 1973):

$$A_{j,k}=A_{j-2,k+1}-\frac{A_{j-2,1}A_{j-1,k+1}}{A_{j-1,1}} \quad (13)$$

$$h_i=\frac{A_{i,1}}{A_{i+1,1}} \quad (14)$$

The reduced transfer function can also be expanded in the form of Eq. (12), i. e.

$$B_{j,k}=B_{j-2,k+1}-\frac{B_{j-2,1}B_{j-1,k+1}}{B_{j-1,1}} \quad (15)$$

$$h'_i=\frac{B_{i,1}}{B_{i+1,1}} \quad (16)$$

Letting the first l coefficient h_i and h'_i of these two series be identical, then the parameters $B_{2,1}$, $B_{2,2}$, $\cdots$,$B_{2,l}$ of the numerator of the reduced model cna be solved from the first l terms of Eqs. (14) and (16).

Example 1

In order to examine the system which has a multiple pole, consider the fifth-order transfer function as follows:

$$G(s)=\frac{2.604s^4+25.046s^3+84.992s^2+118.742s+56.216}{s^5+12s^4+55s^3+120s^2+124s+48}=\sum_{i=1}^{2}\frac{g_i}{(s+p_1)^i}+\sum_{i=3}^{5}\frac{g_i}{s+p_i}. \quad (17)$$

The parameters of Eq. (17) are listed as follows:

$p_1=2$, $g_1=100.0$,
$g_2=0.1$,

$p_3=1, \quad g_3=0.2,$
$p_4=3, \quad g_4=10.0,$
$p_5=4, \quad g_5=20.0.$

The power contribution of each dynamic mode p_i of the transfer function $G(s)$ is given in Table 1. By preserving the dynamic modes, $g_1/s+p_1$ and $g_5/s+p_5$, the denominator can be expressed as

$$R(s)=\frac{B_{2,1}+B_{2,2}s}{8+6s+s^2}. \tag{18}$$

The parameters $B_{2,1}$ and $B_{2,2}$ are obtained from Eqs. (13) through (16). The reduced model is

$$G(s)=\frac{2.613s+9.369}{s^2+6s+8} \tag{19}$$

The unit step responses of the original system and the reduced order model are shown in Fig. 1. From the graph it can be seen that $R(s)$ is good approximation to $G(s)$.

Example 2

In order to examine the system which has multiple complex poles, consider the following sixth-order transfer function:

$$G(s)=\frac{20.5s^5+153s^4+458.2s^3+733.4s^2+633.2s+246.4}{s^6+9s^5+34s^4+72s^3+92s^2+68s+24}$$
$$=\frac{g_1}{s+p_1}+\frac{g_2}{(s+p_1)^2}+\frac{g_3}{s+p_3}+\frac{g_4}{(s+p_3)^2}+\frac{g_5}{s+p_5}+\frac{g_6}{s+p_6}, \tag{20}$$

where

$p_1=-1-i, \quad g_1=5.0,$
$g_2=0.05+0.05i,$
$p_3=-1+i, \quad g_3=5.0,$
$g_4=0.05-0.05i,$
$p_5=2, \quad g_5=10.0,$
$p_6=3, \quad g_6=0.5,$

where p_1, p_3, g_2 and g_4 are complex numbers.

The power contribution of each dynamic mode of the transfer function $G(s)$ is given in Table 2. The power contributions of $g_1/s+p_1$ and $g_3/s+p_3$ forms a complex conjugate.

Also, the power contributions of $g_2/(s+p_1)^2$ and $g_4/(s+p_2)^2$ forms a complex conjugate. Therefore, their power contributions are defined as the sum for each conjugate pair.

From the table, since the power contributions of the dynamic modes $g_1/s+p_1$, $g_3/s+p_3$ and $g_5/s+p_5$ are dominant, the reduced model can be

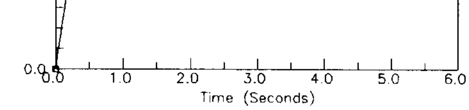

Fig. 1 Unit step responses of the original system and the reduced model

Table 1 Power contribution of each dynamic mode

Dynamic mode	Power contribution
$\frac{g_1}{s+p_1}$	3040.625 (82.004%)
$\frac{g_2}{(s+p_1)^2}$	0.703 (0.019%)
$\frac{g_3}{s+p_3}$	7.989 (0.215%)
$\frac{g_4}{s+p_4}$	245.778 (6.629%)
$\frac{g_5}{s+p_5}$	412.760 (11.132%)

Table 2 Power contribution of each dynamic mode

Dynamic mode	Power contribution
$\frac{g_1}{s+p_1}, \frac{g_3}{s+p_3}$	56.364 (62.614%)
$\frac{g_2}{(s+p_1)^2}, \frac{g_4}{(s+p_3)^2}$	0.334 (0.371%)
$\frac{g_5}{s+p_5}$	31.140 (34.593%)
$\frac{g_6}{s+p_6}$	2.180 (2.422%)

expressed as

$$R(s)=\frac{B_{2,1}+B_{2,2}s+B_{2,3}s^2}{4+6s+4s^2+s^3}. \quad (21)$$

The parameters $B_{2,1}$, $B_{2,2}$ and $B_{2,3}$ are obtained from Eqs.(13) through (16). The reduced model is

$$G(s)=\frac{20.38s^2+50.781s+41.068}{s^3+4s^2+6s+4}. \quad (22)$$

The unit step responses of the original system and the reduced order model are shown in Fig. 2.

Example 3

This example is chosen to compare the responses of the reduced order model obtained by this method with the responses of reduced models obtained by Chen et al. (1980) and Shamash (1975), and to examine the system which has a pole at the origin. Consider the fifth-order transfer function as follows:

$$G(s)=\frac{14.2s^4+94.8s^3+202.2s^2+146.8s+24}{s^5+10s^4+35s^3+50s^2+24s}$$
$$=\frac{1}{s}+G'(s), \quad (23)$$

where

$$G'(s)=\sum_{i=1}^{4}\frac{g_i}{s+p_i} \quad (24)$$

The parameters of Eq. (24) are listed as follows:

$p_1=1$, $g_1=0.2$,
$p_2=2$, $g_2=2.0$,
$p_3=3$, $g_3=1.0$,
$p_4=4$, $g_4=10.0$.

The power contribution of each dynamic mode p_i of the transfer functin $G'(s)$ is given in Table 3.

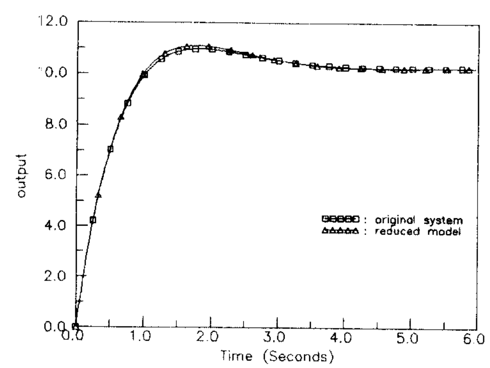

Fig. 2 Unit step responses of the original system and the reduced model

By preserving the dynamic modes corresponding to p_2 and p_4, the denominator of the reduced model can be expressed as

$$R(s)=\frac{B_{2,1}+B_{2,2}s}{8+6s+s^2}+\frac{1}{s}. \quad (25)$$

The parameter $B_{2,1}$ and $B_{2,2}$ can be solved from Eqs. (13) through (16). The reduced model is

$$R(s)=\frac{13.2s+32.266}{s^2+6s+8}+\frac{1}{s}$$
$$=\frac{14.2s^2+38.266s+8}{s^3+6s^2+8s}. \quad (26)$$

The reduced order models of the same original system as that treated by Chen et al. and Shamash are given as follows:

$$R(s)=\frac{4.87459s+2.82213}{s^2+1.4577s+0.6997}+\frac{1}{s}$$
(by Chen et al.)
$$R(s)=\frac{9.228s+8.067}{s^2+3s+2}+\frac{1}{s}$$
(by Shamash)

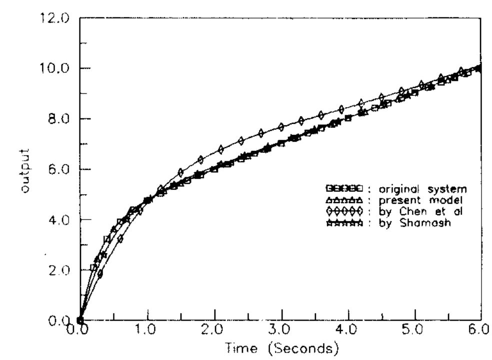

Fig. 3 Unit step responses of the original system and the reduced models

Table 3 Power contribution of each dynamic mode

Dynamic mode	Power contribution
$\frac{g_1}{s+p_1}$	0.603 (2.396%)
$\frac{g_2}{s+p_2}$	4.867 (19.330%)
$\frac{g_3}{s+p_3}$	2.045 (8.123%)
$\frac{g_4}{s+p_4}$	17.662 (70.151%)

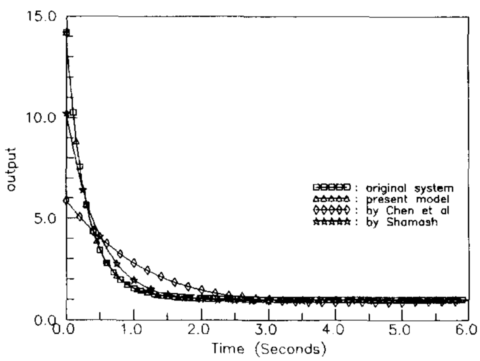

Fig. 4 Unit impulse responses of the original system and the reduced models

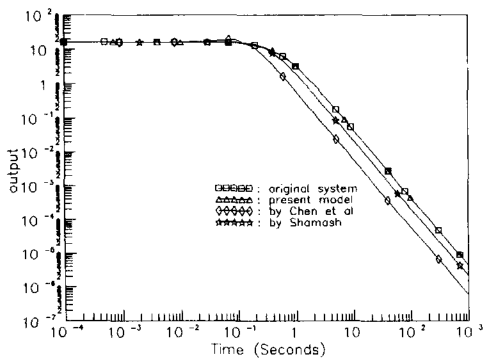

Fig. 5 Spectral density functions of the original system and the reduced models

The unit step responses of the original system and the reduced order model of this method and other methods (by Chen et al. and Shamash) and their impulse responses are shown in Figs. 3 and 4, respectively. Also, their power spectral density function $G'_{yy}(f)$ of which the transfer function $G'(s)$ is generated by removing the term $1/s$, is shown in Fig. 5 where input signals are assumed to be white noises. It can be said that the present method gives better approximation to the orginal system than other methods.

4. Conclusion

The method of model reduction based on dispersion analysis and the continued fraction is extended to treat the system which has multiple poles, or has simple or multiple poles on the imaginary axis. The power contribution based on power spectral density function is used for order reduction. By discarding the dynamic modes with small power contributions, the denominator of the reduced model is obtained. The continued fraction method is used to determine the numerator of the reduced model.

Since the power contribution of each dynamic mode is easily obtained through arithmetic calculation, it is computationally easy to program. Through examples, this method is known to give better approximation to the original system than other methods and to be able to treat the system which has multiple poles, and simple or multiple poles on the imagniary aixs.

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